

A Multi-Resonator Network for Superconducting Circuits

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Superconducting circuits have emerged as a configurable and coherent system to investigate a wide variety of quantum behaviour[1]. This architecture — circuit QED — has been used to demonstrate phenomena from quantum optics[2], quantum limited amplification[3], and small-scale quantum computing[4]. There is broad interest in expanding circuit QED to simulate lattice models (e.g., the Jaynes-Cummings-Hubbard model), generate long-distance entanglement, explore multimode quantum optics, and for topological quantum computing[5]. Here we introduce a new multi-resonator (multi-pole) circuit QED architecture where qubits interact through a network of strongly coupled resonators. This circuit architecture is a novel system to study multimode quantum optics[6], quantum simulation[5, 7], and for quantum computing. In this work, we show that the multi-pole architecture exponentially improves contrast for two-qubit gates without sacrificing speed, addressing a growing challenge as superconducting circuits become more complex. We demonstrate the essential characteristics of the multi-pole architecture by implementing a three-pole (three-resonator) filter using planar compact resonators[8], which couples two transmon-type[9] qubits. Using this setup we spectroscopically confirm the multimode circuit QED model, demonstrate suppressed interactions off-resonance, and load single photons into the filter. Furthermore, we introduce an adiabatic multi-pole (AMP) gate protocol to realize a controlled-Z gate between the qubits and create a Bell state with 94.7% fidelity.

Josephson junction (JJ) qubits have demonstrated the essential building blocks of gate based quantum computing. Single qubit coherence times can approach 100 μ s [10], arbitrary rotations in the Bloch sphere are possible with gate fidelities higher than 99.8%[11], and elementary two-qubit gates have attained gate fidelities up to 96.2%[12]. There has been rapid progress towards constructing larger circuits to implement quantum algorithms[12–15], photonic memories[16, 17], and nascent quantum simulation[18].

As circuits grow larger, interactions must be carefully controlled. To counteract “always on” interactions in

NMR quantum computing, decoupling pulse sequences have been developed[19]. These sequences can be applied to JJ qubits, but become onerous as the system size grows larger. Instead, a more robust approach is to develop tunable interactions for high contrast gates, most commonly by coupling JJ qubits through a resonant interaction. In these experiments, interactions are controlled via the detuning from resonance, imposing a tradeoff between gate contrast and speed. Also, expanding beyond two qubits results in spectral crowding, which limits addressability[20] and introduces spurious avoided crossings. While parametric gates[21] sidestep some of these problems, in both cases the contrast is only linear in detuning. This limits the achievable off-rate since detuning is bounded. Alternatively, we can dynamically tune the coupling by destructive interference between two charge qubits[22] or by flux tuning a JJ inductive coupler[23]. However, there are limits to dynamic coupling since additional junctions are required, which introduces complexity and a new path for decoherence.

Here, instead of coupling through a single resonator, we couple through a linear chain of strongly coupled resonators, analogous to a multi-pole bandpass filter. The multi-pole architecture enables the off-resonant interactions to be suppressed exponentially in the number of resonators (poles) without any additional active elements. Further, although not explored in this work, the strongly coupled resonator network can be employed as a lattice for quantum simulation, and a photonic register for quantum memory. This multi-pole architecture could also be utilized to filter the qubit noise environment, i.e. a multi-pole Purcell filter[25, 26].

A schematic of our circuit and the corresponding physical realization are illustrated in Fig. 1. Three identical resonators of frequency ω_F are capacitively coupled to each other in a chain to form our quantum filter. Two flux-tunable transmon qubits ($\omega_Q \approx 2\pi \times 1 - 9$ GHz) are capacitively coupled to the resonators at the end of the filters. For qubit frequencies $\omega_{Q,1}$, $\omega_{Q,2}$, the qubit-filter

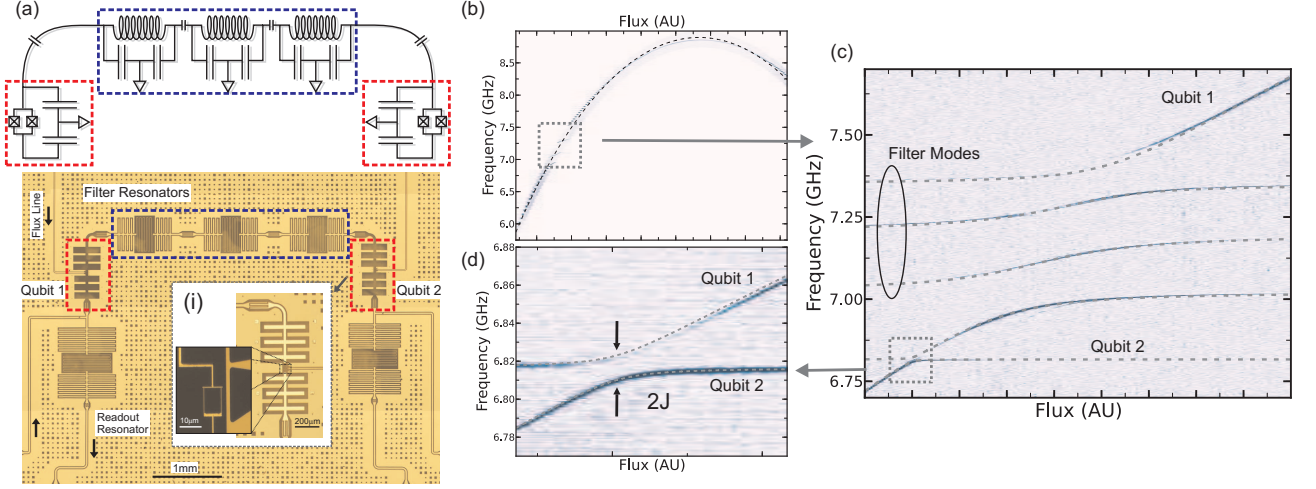


FIG. 1: Multi-pole device schematic and spectroscopy. (a) Schematic (top) and optical image (bottom) of our 3-pole cavity QED device. The schematic shows the three lumped LC resonators of the filter (blue dotted line) that couple two transmon-type qubits (red dotted line). These same features are outlined on the physical circuit. All resonators and large features are Nb (light yellow) etched on sapphire (dark). The qubits have not been deposited in the largest image, but are illustrated after aluminum deposition in the inset (i). Transmission lines from the top-left/top-right allow for fast-flux biasing of the transmon. The qubits are also coupled to readout resonators at $\omega_{1(2)} = 2\pi \times 4.20(4.65)$ GHz and the qubit state is inferred by measuring the transmission of these resonators. The qubit 1 (qubit 2) lifetime $T_1 = 2.36(2.14)\mu\text{s}$ and the decay of Ramsey coherence (fit to Gaussian decay $e^{-t^2/2\sigma^2}$) is $\sigma = 312(492)\text{ns}$. Full fabrication details, qubit properties, instrumentation, and cryogenic setup are given in [24]. (b) Single qubit spectroscopy as the qubit energy is tuned using the flux line. The dashed line is a fit obtained by diagonalizing the energy levels of the transmon in the charge basis. (c) Spectroscopy of the region where the qubit energy crosses through the filter modes (dashed box in (b)). The energy of the other qubit is fixed and below the filter. The dashed lines are the eigenvalues of the Hamiltonian given by Eqn. 1 using the qubit-filter parameters listed in the main text. (d) Spectroscopy of the qubit-qubit avoided crossing (dashed box in (c)). The minimum energy separation gives twice the exchange energy $2J$ (Eqn. 5).

system (for n -poles) is described by the Hamiltonian

$$\hat{H} = \hat{H}_Q + \hat{H}_F + \hat{H}_{Q-F} \quad (1)$$

$$\hat{H}_Q = \omega_{Q,1}\hat{\sigma}_1^Z/2 + \omega_{Q,2}\hat{\sigma}_2^Z/2 \quad (2)$$

$$\hat{H}_F = \sum_{i=1}^n \omega_F \hat{a}_i^\dagger \hat{a}_i + \sum_{i=2}^n g_F (\hat{a}_i^\dagger \hat{a}_{i-1} + \hat{a}_{i-1}^\dagger \hat{a}_i) \quad (3)$$

$$\hat{H}_{Q-F} = g_{Q1,F} (\hat{a}_1^\dagger \hat{\sigma}_1^- + \hat{a}_1 \hat{\sigma}_1^+) + g_{Q2,F} (\hat{a}_n^\dagger \hat{\sigma}_2^- + \hat{a}_n \hat{\sigma}_2^+) \quad (4)$$

where \hat{a}_i^\dagger creates a photon in the i^{th} resonator, $\hat{\sigma}^{+(-)}$ is the raising (lowering) operator for the qubit, $\hat{\sigma}^Z$ is the z-Pauli operator, g_F is the filter-filter coupling and $g_{Q,F}$ is the qubit-filter coupling. For clarity we omit higher transmon levels; these are fully considered in Ref. [27]. Strong coupling between the bare filter resonators splits the three degenerate resonators into three “filter” modes with energies $\omega = \omega_F, \omega_F \pm \sqrt{2}g_F$. Each of these filter modes are a superposition of photons in the bare resonators, i.e., starting from one photon in each of the bare filters $|100\rangle = \{1, 0, 0\}, |010\rangle = \{0, 1, 0\}, |001\rangle = \{0, 0, 1\}$ the coupling transforms these into the states $|\omega_F - \sqrt{2}g_F\rangle = \{\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\}, |\omega_F\rangle = \{\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\}, |\omega_F + \sqrt{2}g_F\rangle = \{\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\}$. Crucially, every filter mode has non-zero weight in the resonators at either end of the

chain so that filter photons couple to both qubits. We fit the spectroscopy data in Fig. 1 (c) to extract qubit-filter parameters $\omega_F = 2\pi \times 7.169\text{GHz}$, $g_F = 2\pi \times 118\text{MHz}$, and $g_{Q1,F}(g_{Q2,F}) = 2\pi \times 135(144)\text{MHz}$.

When the qubits are detuned from all the filter modes and the filter is empty (analogous to the stop band of a classical filter) residual interactions are mediated by virtual photons through all modes and we can rewrite Eqn. 1 as

$$\hat{H} = \hat{H}_Q + J (\hat{\sigma}_1^+ \otimes \hat{\sigma}_2^- + \hat{\sigma}_1^- \otimes \hat{\sigma}_2^+) + \xi \hat{\sigma}_1^Z \otimes \hat{\sigma}_2^Z, \quad (5)$$

where J is the exchange term and ξ is the controlled phase (c-phase) rate. If we consider identical qubit 1 and qubit 2 filter couplings g_Q and let Δ be the averaged detuning of the qubit from the bare filter mode (i.e., $\Delta = (\omega_{Q1} + \omega_{Q2} - 2\omega_F)/2$), then we can approximate J and ξ (for an n -pole filter) as

$$J \approx \frac{g_Q^2}{g_F} \left(\frac{g_F}{\Delta} \right)^n, \quad (6)$$

$$\xi \approx \frac{4nJ^2}{\Delta}. \quad (7)$$

Notably, these rates are suppressed exponentially in the number of filter modes n , in terms of the small parameter g_F/Δ . This is a result of destructive interference

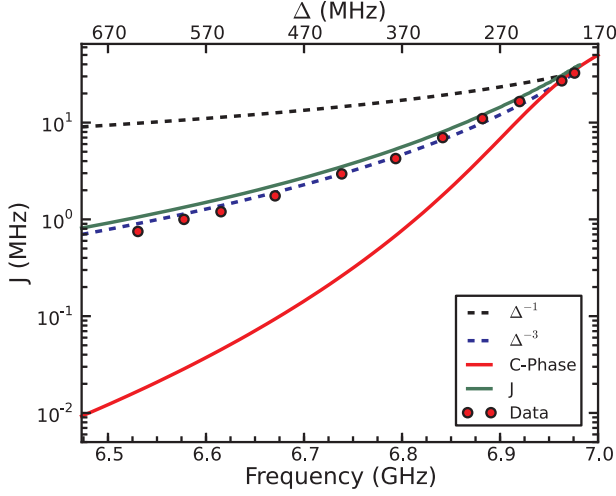


FIG. 2: **Off-resonant coupling.** Qubit-qubit exchange rate J as a function of the qubit frequency (top axis, detuning Δ from the bare cavity frequency) for two different scaling laws (dashed lines), by numerical diagonalizing Eqn. 1 (green line), and by measuring the exchange splitting (data points). To measure the exchange splitting we analyze the qubit spectroscopy as the flux bias is tuned so that the two qubits cross in frequency, sample data is shown in Fig. 1. The splitting at the avoided crossing measures $2J$. We also plot a numerical calculation of the c-phase (from Eqn. 1) versus the qubit 1 frequency using the filter parameters determined by the fit in Fig. 1 where qubit 2 is detuned below qubit 1 by 50 MHz (red line).

between multiple filter modes which sum coherently and enables the ability to turn off interactions with high contrast by detuning from the filter. To confirm the off-rate scaling predicted by Eqns. 6 and 7, we directly measure the exchange term J from qubit spectroscopy, and numerically calculate the c-phase rate. The data plotted in Fig. 2 agrees well to the model with no free parameters, demonstrating the essential scaling of the multi-pole off-rate, and implying an off-rate less than 10kHz for a qubit-qubit detuning of 50MHz.

To enable strong interactions in the multi-pole architecture we tune the qubit frequency into resonance. In this limit, the expressions for the suppressed couplings given by Eqns. 6 and 7 are invalid because the virtual photon paths no longer destructively interfere. Since the qubit interacts primarily with the closest mode, the coupling strength is of order g . Interestingly, if there is a real photon in one of the filter modes the interference is also imbalanced, hence the Stark shift is not suppressed. In both cases, the qubits interact predominantly through a single mode and so the strong interaction physics is essentially that of two qubits interacting through a single cavity.

For our controlled-Z AMP gate, we utilize these strong interactions by loading a real photon into the lowest filter mode and then employing a state-dependent one-photon

Stark shift. To load a single photon we adiabatically traverse the qubit-filter avoided crossing shown in Fig 1. To study the dynamics of this crossing, we excite the qubit, ramp through the filter and back, and then measure the residual qubit population (Fig. 3 (a)). Because we traverse avoided crossings twice, we observe interference fringes. There are two types of fringes in Fig 3(a): fast fringes at short times and slower fringes dominant at longer times. The fast fringes correspond to ramp speeds larger than the total filter bandwidth ($\gtrsim 400\text{MHz}$) where a significant fraction of the excitation remains with the qubit[28]. The slower fringes correspond to the excitation being distributed over multiple filter modes and the fringe frequency is fixed by the filter mode splitting. The multimode nature of the crossing is advantageous; although the ramp is not adiabatic with the lowest filter mode unless it is slower than $\approx 25\text{ns}$, the excitation remains in the filter for ramps $> 5\text{ns}$. We exploit this multimode Landau-Zener physics to transfer population to the filter faster than the single mode adiabatic limit.

Next, we measure the interaction between a single photon and a qubit in the ground state by performing a Ramsey experiment on the photon while varying the length of the interaction at different detunings (Fig. 3 (b)). This protocol, which essentially measures the one-photon Stark shift, has the advantage of being fixed time for the photon state. We prepare qubit 1 in the superposition state $(|g\rangle + |e\rangle)/\sqrt{2}$ and ramp the qubit through the filter to coherently load a superposition of zero and one photon in the filter (predominantly in the lowest filter mode). Once the filter state has been prepared, we dynamically change the detuning of qubit 2 with respect to the filter and hold for time τ . The filter state is then converted back to a qubit 1 excitation and measured. This process measures the phase accumulated by qubit 1 in the frame rotating at the qubit starting energy. There is a large single qubit phase which is held constant, so the Ramsey fringe frequency only measures the interaction shift. Approaching the filter from below, the Stark shift increases as $\approx 1/\Delta$, and then saturates at the maximum interaction (approximately the filter splitting $\sqrt{2}g_F = 167\text{MHz}$) as qubit 2 is brought through the filter.

Finally, we combine these capabilities to construct a quantum logic gate. As in the previous experiment, to build the gate we convert the qubit 1 excitation into a photon, move qubit 2 close to the filter to acquire a state dependent Stark shift, and then return the photon back to qubit 1. The state populations remain fixed and each of the four computational states acquires a phase $(\phi_{|gg\rangle}, \phi_{|eg\rangle}, \phi_{|ge\rangle}, \phi_{|ee\rangle})$. The conditional phase is $\phi_{\text{c-phase}} = \phi_{|ee\rangle} + \phi_{|gg\rangle} - (\phi_{|eg\rangle} + \phi_{|ge\rangle})$ which we calculate in Fig. 4 (b). A controlled-Z gate (CZ) is realized when the conditional phase is π . The qubits will also acquire trivial single qubit phases which we calibrate out by fine-tuning the flux pulse shape. Although the largest one-photon Stark shifts occurs when we bring qubit 2

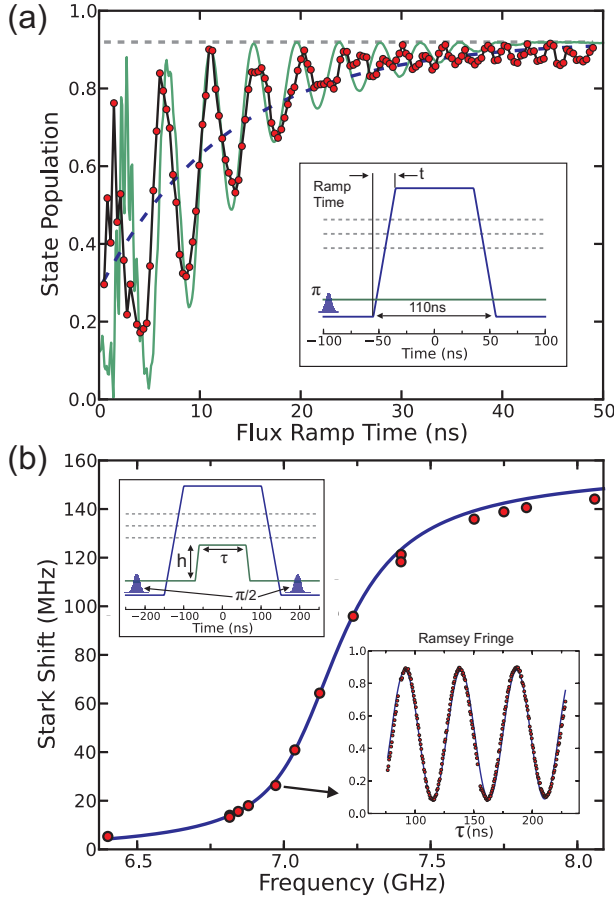


FIG. 3: Single photon loading and Stark shift. (a) Loading a single photon into the filter. First, qubit 1 is excited and then the qubit energy is raised linearly through the filter in time t , held for time $T - 2t$, ramped back in time t and measured. The total time T is fixed to 110ns (see flux diagram inset). We plot the qubit state versus the ramp time: the solid black line is a guide to the eye, the dashed blue line is an exponential fit to show the general adiabatic trend, the dashed grey line is the expected maximum state population given T_1 decay, and the green solid line is a numerical solution of the Schrodinger equation using the Hamiltonian given by Eqn. 1 and scaled by T_1 decay. (b) Measurement of the the one-photon Stark shift versus the numerical calculation (blue solid line) from Eqn. 1 using the parameters from the main text and assuming the photon is in the lowest filter mode. The Stark shift is measured from the Ramsey fringes versus τ ; sample data for one of the points is shown as an inset. The timing diagram of the flux pulses used for this measurement is inset. Gaussian pulses indicate $\frac{\pi}{2}$ pulses applied before and after the flux pulse. Qubit 1 (blue solid line) is raised through the filter in 25ns and held for a fixed amount of time. Qubit 2 (green solid line) is raised to a height h for time τ . We use 5.3GHz as the reference height, and so set the Stark shift at that point to zero (the true Stark shift is referenced to $\Delta \rightarrow \infty$).

through the filter (Fig. 3 (b)), the largest state-dependent interaction occurs when we bring qubit 2 just below the filter because of the full state structure of the transmon

(see discussion in Ref. [27]). The flux pulse sequence for our CZ gate — the adiabatic multi-pole (AMP) gate — is illustrated in Fig. 4. The total gate time, 95ns, is optimized to maximize gate fidelity. For 50MHz detuning between the qubits, this implies a gate contrast (on/off rate) greater than 1000 even for relatively small $\Delta/g \approx 5$.

To demonstrate the AMP gate we prepare a Bell state by applying single qubit pulses before and after the gate. Ideally this process creates the Bell state $|\Psi_{Bell}\rangle = (|gg\rangle + e^{i\Phi}|ee\rangle)/\sqrt{2}$. To characterize the expected density matrix we perform state tomography[29] on both qubits after the gate (see Fig. 4 (c)). The fidelity $F = \langle \Psi_{Bell} | \rho_{meas} | \Psi_{Bell} \rangle$ is $0.947 \pm 0.005_{stat} \pm 0.01_{sys}$ corresponding to concurrence of $0.926 \pm 0.01_{stat} \pm 0.02_{sys}$ [30]. We also measure a full process fidelity of 0.89 (errors and tomography details are discussed in [24]). Our fidelity is comparable to other contemporary results (two-qubit entangled states have been produced with state fidelities up to 98.3% and concurrence of 0.994 [31]), and is limited by our lifetime, rather than the protocol. One advantage of our protocol is that our gate is relatively insensitive to inhomogeneous broadening due to flux noise since the filter is not flux tunable. Several improvements are possible, for example, engineering a flux insensitive bias point below the filter for state preparation [32], utilizing new materials [33] and material processing for high Q resonators [34], as well as reducing the total gate time using techniques from optimal control for crossing the filter.

In conclusion, we have demonstrated a new multi-pole architecture for coupling superconducting qubits. We measured that the off-resonance coupling is suppressed exponentially in the number of poles, while still maintaining strong interactions when the qubits are tuned close to resonance. We used these capabilities to realize a high-contrast controlled-Z (AMP) gate. Further, this work indicates a need to develop a microwave filter theory for coherent quantum systems. The multi-pole architecture is a promising platform for realizing lattice based quantum simulations and photonic registers for quantum information processing.

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Author Contributions: DM and PR designed and fabricated the sample. DM, RN and PR acquired the data which was analyzed by DM. DM and DS wrote the manuscript with feedback from all authors. LB provided theoretical support. DM, LB and DS outlined the experiment. DS supervised all aspects of the project.

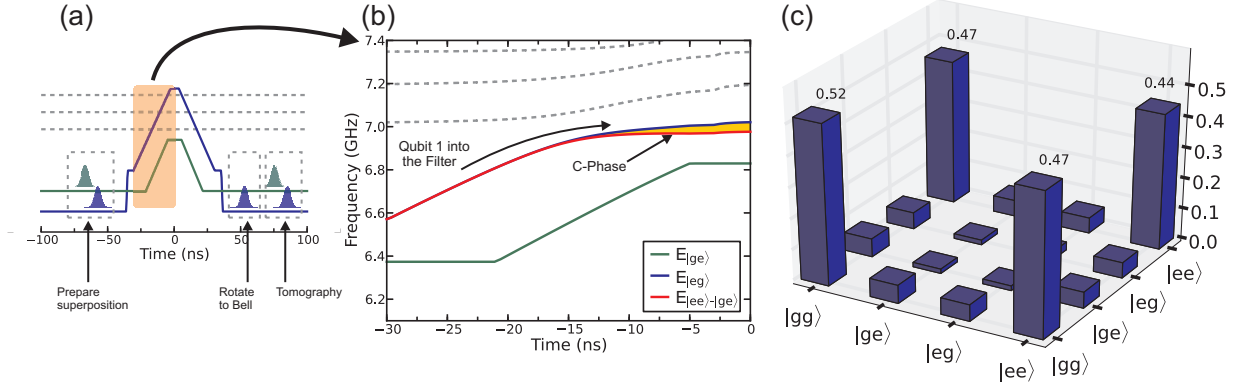


FIG. 4: **Bell State** (a) Flux pulse (solid lines) and microwave pulse (Gaussian) timing diagram to create a Bell state using our multi-pole filter and perform state tomography. We first apply $\frac{\pi}{2}$ pulses to qubit 1 (blue) and qubit 2 (green). All microwave pulses are applied to the qubits simultaneously, but are offset in the diagram for clarity. Then, we ramp qubit 1 through the filter, and after a short delay ramp qubit 2 close to the filter. We reverse the process and apply a $\frac{\pi}{2}$ pulse to qubit 1. Tomography pulses are then applied to both qubits and we independently measure the state of both qubits. (b) The energy levels calculated from Eqn. 1 in the orange region highlighted in (a). As qubit 1 approaches the filter there is an avoided crossing with the first filter mode which converts the qubit excitation into a filter photon. When qubit 2 is raised, the energy of the filter photon depends on the state of qubit 2, which generates a c-phase. The total phase is the yellow area indicated in the graph. (c) Absolute value of the density matrix elements after state tomography of the Bell state produced by the gate. Details of the state tomography are given in [24] and fidelities are discussed in the main text.

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